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On dynamic weighted survival entropy of order α

G.Rajesh[*], E.I.Abdul-Sathar and Rohini, S. Nair[†]

Abstract: Recently Abasnejad et al. (2010) proposed a measure of uncertainty based on survival function, called the survival entropy of order α. A dynamic form of the survival entropy of order α is also proposed by them. In this paper, we derive the weighted form of these measures. The properties of the new measures are also discussed.

Keywords: life distributions, survival entropy, Ordering, Reliability

[*]Department of Statistics, DB Pampa College, Parumala-689 626, India.
[†]Department of Statistics, University of Kerala, Thiruvananthapuram - 695 581, India

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1 Introduction

In the recent past, many researchers have shown keen interest in the measurement of uncertainty associated with a probability distribution. Of particular interest in probability and statistics is the notion of entropy, introduced by Shannon (1948). If X is a random variable having an absolutely continuous distribution function F with probability density function f, then the entropy of the random variable X is defined as

$$H(X) = -\int_0^{+\infty} f(x)\log f(x)\,dx = -\mathbf{E}[\ln \mathbf{f}(\mathbf{X})]. \tag{1}$$

Observing that if a unit has survived up to time $\boldsymbol{t}$, $H(X)$ is not a useful tool for measuring the uncertainty about the remaining lifetime of the unit, Ebrahimi and Pellery (1995) and Ebrahimi (1996) proposed the concept of residual entropy in terms of a conditional measure. For a non-negative random variable X representing the lifetime of a component, the residual entropy function is the Shannon's entropy associated with the random variable $X|X > t$, and is defined as

$$H(X,t) = -\int_{\mathbf{t}}^{\infty} \frac{f(x)}{\bar{F}(t)}\log\frac{f(x)}{\bar{F}(t)}\,dx, \tag{2}$$

where $\bar{F}(t) = P(X > t)$ denotes the survival function. For a discussion on properties of $H(X,t)$ we refer to Nair and Rajesh (1998), Asadi and Ebrahimi (2000) and Asadi et al. (2004). Even though Shannon's entropy finds applications in many areas of research, recently, Rao et al. (2004) identified some limitations of the use of (1) in measuring the randomness of certain systems and introduced an alternative measure of uncertainty called cumulative residual entropy (CRE), through

$$\varepsilon(X) = -\int_0^{\infty} \bar{F}(x)\log\bar{F}(x)\,dx. \tag{3}$$

Recently, Asadi and Zohrevand (2007) introduced the concept of CRE in terms of a conditional measure called dynamic cumulative residual entropy (DCRE). For a non-negative random **variable** X **representing the lifetime** of a component, the dynamic cumulative residual entropy (DCRE) is the CRE associated with the random variable $X|X > t$, and is defined as

$$\varepsilon(X;t) = -\int_t^{\infty} \frac{\bar{F}(x)}{\bar{F}(t)}\log\frac{\bar{F}(x)}{\bar{F}(t)}\,dx. \tag{4}$$

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For a discussion on the properties and applications of the DCRE we refer to Asadi and Zohrevand (2007), Di Crescenzo and Longobardi (2009) and Navarro et al.(2010).

The concept of weighted distribution is usually considered in connection with modelling statistical data, where the usual practise of employing standard distributions is not found appropriate in some cases. In recent years, this concept has been applied in many areas of statistics, such as analysis of family size, human heredity, world life population study, renewal theory, biomedical, statistical ecology, reliability modelling etc. Associated to a random variable X with probability density function f and to a non-negative real function w, we can define the weighted random variable X^w with density function.

$$f^w(x) = \frac{w(x)f(x)}{E(w(x))},$$

where we assume $0 < E(w(x)) < \infty$. When $w(x) = x$, X^w is called the length (or size) biased random variable and it is denoted by X^*.

For recent works on weighted distributions, we refer to Navarro et al. (2001), **Nair and Sunoj (2003)**, Pakes et al. (2003), Di Crescenzo and Longobardi (2006), Oluyede and Terbeche (2007) and Sunoj and Maya (2008).

Belis and Guiasu (1968) introduced the concept of weighted entropy through

$$\begin{aligned} H^w(X) &= -\int_0^{+\infty} x f(x) \log f(x)\, dx \\ &= -\int_0^{+\infty} dy \int_y^{\infty} f(x) \log f(x)\, dx. \end{aligned} \tag{5}$$

The factor x, inside the integral on the right hand side of (5) represents a weight linearly emphasizing the occurrence of the event $\{X = x\}$. This yields a length biased shift dependent information measure assigning greater importance to larger values of X. For more properties and applications of weighted entropy, we refer to Belis and Gaiasu (1968) and Longo (1976).

In order to introduce a new shift-dependent dynamic information measures, **Di Crescenzo and Longobardi (2006) introduced** a weighted entropy for residual lifetime, that is the weighted version of (2). The weighted residual entropy of X at time t is defined as

$$H^w(X;t) = -\int_t^\infty x\, \frac{f(x)}{\bar{F}(t)} \log \frac{f(x)}{\bar{F}(t)}\, dx. \tag{6}$$

The entropy (6) has been used to characterize lifetime distributions. Some results on the decomposition of weighted entropy, an ordering of lifetime distributions and on the monotonic transformation of (6) is discussed in Di Crescenzo and Longobardi (2006).

Recently Suchismita Das (2013) introduced the concept of weighted generalized entropy and is given by

$$\mathbf{H_1^{w^\beta}(X;t)} = \frac{\mathbf{1}}{\beta - \mathbf{1}}\left[\mathbf{1} - \int_{\mathbf{t}}^\infty \left(\frac{\mathbf{f^w(x)}}{\mathbf{\bar{F}^w(t)}}\right)^\beta \mathbf{dx}\right], \tag{7}$$

and

$$\mathbf{H_2^{w^\beta}(X;t)} = \frac{\mathbf{1}}{\mathbf{1} - \beta} \ln \int_{\mathbf{t}}^\infty \left(\frac{\mathbf{f^w(x)}}{\mathbf{\bar{F}^w(t)}}\right)^\beta \mathbf{dx}, \tag{8}$$

where $f^w(t)$ **and** $\bar{F}^w(t)$ **are the probability density function and survival function of a length based weighted random variable** X_w **associated to the random variable** X**, and is given by**

$$\mathbf{f^w(t)} = \frac{\mathbf{t}}{\mathbf{E(X)}}\mathbf{f_X(t)} \quad \text{and} \quad \mathbf{\bar{F}^w(t)} = \frac{\mathbf{E(X|X > t)}}{\mathbf{E(X)}}\mathbf{\bar{F}_X(t)}.$$

Recently, Misagh et al. (2011) introduced a measure of uncertainty, called the **weighted cumulative residual entropy (WCRE) and studied various properties of it, is defined as**

$$\varepsilon^w(X) = -\int_0^\infty x\, \bar{F}(x) \log \bar{F}(x)\, dx. \tag{9}$$

Recently, Abasnejad et al. (2010) proposed a measure of uncertainty based on survival function, called the survival entropy of order α (SE), and is defined as

$$\varepsilon_\alpha(X) = -\frac{1}{\alpha - 1} \log \int_0^\infty \bar{F}^\alpha(x)\, dx \quad \forall \alpha > 0 \quad (\alpha \neq 1). \tag{10}$$

In the present work, **we derive the weighted version** of SE defined in (10) and it is termed as weighted survival entropy of order α (WSE) and study various properties of this measure. The rest of the paper is organized as follows. Section 2 gives some properties of the WSE. In Section 3, we introduce dynamic form of the WSE called dynamic weighted survival entropy (DWSE) and give some bounds for it based on weighted mean residual lifetime (WMRL). Section 4 gives a result that relates the DWSE and the hazard rate ordering. Finally Section 5 gives the characterization of Rayleigh distribution using DWSE.

2 Weighted survival entropy properties

Let X be a non-negative random variable having an absolutely continuous distribution function $F(x)$, probability density function $f(x)$, then weighted survival entropy of order α of the random variable X is defined as

$$\varepsilon_{\alpha}^{w}(X) = \frac{1}{1-\alpha} \log \int_{0}^{\infty} x\bar{F}^{\alpha}(x)dx \quad \forall \alpha > 0 \ (\alpha \neq 1). \tag{11}$$

(11) can also be written as

$$\begin{aligned} \exp\{(1-\alpha)\varepsilon_{\alpha}^{w}(X)\} &= \int_{0}^{\infty} x\,\bar{F}^{\alpha}(x)dx. \\ &= \frac{1}{\alpha}\int_{0}^{\infty} \bar{F}^{\alpha}(x)m(x)\,dx - \frac{\alpha-1}{\alpha}\int_{0}^{\infty} x\bar{F}^{\alpha}(x)m'(x)\,dx, \end{aligned}$$

where $m(x) = E(X - x|X > x)$, is the mean residual life function (MRL) and $m'(x)$ is the first derivative of $m(x)$ with respect to x. The following example shows that although two distributions have the same SE but they have different WSE.

Example 2.1: Let X and Y be random variables with density functions

$$f_X(x) = \begin{cases} \frac{1}{2}, & 2 \leq X \leq 4 \\ 0, & otherwise. \end{cases}$$

and

$$f_Y(y) = \begin{cases} \frac{1}{2}, & 0 \leq Y \leq 2 \\ 0, & otherwise. \end{cases}$$

Then we can see that

$$\varepsilon_\alpha(X) = \varepsilon_\alpha(Y) = \frac{1}{1-\alpha}\log\left(\frac{2}{1+\alpha}\right).$$

where $\varepsilon_\alpha(X)$ and $\varepsilon_\alpha(Y)$ are the SE of X and Y. The WSE of the random variables X and Y are given by

$$\varepsilon^w_\alpha(X) = \frac{1}{1-\alpha}\log\left(\frac{4(3+\alpha)}{(\alpha+1)(\alpha+2)}\right).$$

and

$$\varepsilon^w_\alpha(Y) = \frac{1}{1-\alpha}\log\left(\frac{4}{(\alpha+1)(\alpha+2)}\right) \quad \text{respectively.}$$

Therefore $\varepsilon^w_\alpha(X) \neq \varepsilon^w_\alpha(Y)$. Hence, even though $\varepsilon_\alpha(X) = \varepsilon_\alpha(Y)$, when $\alpha > 1 (0 < \alpha < 1)$, the WSE about the predictability of X by the density function $f_X(x)$ is smaller (greater) than the predictability of Y by the density function $f_Y(y)$.

The following lemma gives the properties of $\varepsilon^w_\alpha(X)$. It shows that $\varepsilon^w_\alpha(X)$ is a shift-independent measure.

Lemma 2.1. *If Y=aX+b, with a>0 and b≥ 0, then*

$$\exp\{(1-\alpha)\varepsilon^w_\alpha(Y)\} = a^2\exp\{(1-\alpha)\varepsilon^w_\alpha(X)\} + ab\,\exp\{(1-\alpha)\varepsilon_\alpha(X)\}. \tag{12}$$

Proof. The proof follows by using the property of survival function, $\bar{F}_{aX+b}(X) = \bar{F}_X(\frac{x-b}{a})$, $X \epsilon R$ and also using (11). □

Corollary 2.1. *From Lemma 2.1, we have*

(i) If b=0 in (12), we get

$$\exp\{(1-\alpha)\varepsilon^w_\alpha(Y)\} = a^2\exp\{(1-\alpha)\varepsilon^w_\alpha(X)\}.$$

(ii) If a=1 in (12), we get

$$\exp\{(1-\alpha)\varepsilon_\alpha^w(Y)\} = \exp\{(1-\alpha)\varepsilon_\alpha^w(X)\} + b\ \exp\{(1-\alpha)\varepsilon_\alpha(X)\}.$$

If $\bar{F}_\theta^*(x)$ and $\bar{F}(x)$ denotes the survival functions of the random variables X_θ^* and X respectively, then the proportional hazard model is described by the relation $\bar{F}_\theta^*(x) = [\bar{F}(x)]^\theta$, where θ is a real number and in Lemma 2.2 we compare the WSE of X, X_θ^* and θX. The proof is omitted.

Lemma 2.2. *For the WSE, the following properties hold:*

(a) $\varepsilon_\alpha^w(X_\theta^*) = \frac{1-\theta\alpha}{1-\alpha}\varepsilon_{\alpha\theta}^w(X)$.

(b) $\varepsilon_\alpha^w(X_\theta^*) \geq \varepsilon_\alpha^w(X) \geq \varepsilon_\alpha^w(X\theta)$, *for* $0 < \theta \leq 1, 0 < \alpha < 1$ *and* $\theta \geq 1, \alpha > 1$.

(c) $\varepsilon_\alpha^w(X_\theta^*) \leq \varepsilon_\alpha^w(X) \leq \varepsilon_\alpha^w(X\theta)$, *for* $0 < \theta \leq 1, \alpha > 1$ *and* $\theta \geq 1, 0 < \alpha < 1$.

Table 1 gives the expressions for $\varepsilon_\alpha^w(X)$, $\varepsilon_\alpha^w(X\theta)$ and $\varepsilon_\alpha^w(X_\theta^*)$ using Uniform on (0,a) and Exponential(λ) distributions.

From Table 1, it can be easily verify that

(i) $\varepsilon_\alpha^w(X_\theta^*) \geq \varepsilon_\alpha^w(X) \geq \varepsilon_\alpha^w(X\theta)$, for $0 < \theta \leq 1, 0 < \alpha < 1$ and $\theta \geq 1, \alpha > 1$.

(ii) $\varepsilon_\alpha^w(X_\theta^*) \leq \varepsilon_\alpha^w(X) \leq \varepsilon_\alpha^w(X\theta)$, for $0 < \theta \leq 1, \alpha > 1$ and $\theta \geq 1, 0 < \alpha < 1$.

Let X be a continuous non-negative random variable with survival function $\bar{F}(x)$ and density function $f(x)$ then the weighted mean residual lifetime (WMRL) of X is defined as

$$m_F^*(t) = \frac{1}{\bar{F}(t)}\int_t^\infty x\bar{F}(x)dx \qquad (13)$$

and $m_F^*(0) = \int_0^\infty x\bar{F}(x)dx$. In the following theorem, we give some bounds for the WSE using $m_F^*(0)$ and $m_F^*(t)$.

Theorem 2.1. *Let X be a non-negative continuous random variable with WMRL* $m_F^*(t)$, *WSE* $\varepsilon_\alpha^w(X)$ *and WCRE* $\varepsilon^w(X)$, *then* *(a)* $\varepsilon^w_\alpha(X) \geq \frac{1}{1-\alpha}\log m^*_F(0)$*, for* $\alpha > 0$.

(b) $\varepsilon^w_\alpha(X) \leq (\geq)\varepsilon^w(X), \forall\alpha > 1\ (0 < \alpha < 1)$.

Proof. We know that (a) $x\bar{F}^\alpha(x) \leq (\geq)\ x\bar{F}(x), \forall\alpha > 1\ (0 < \alpha < 1)$.
Integrating both sides of the above equation with respect to x, the result follows. (b) It can be easily shown that

$$\varepsilon^w_\alpha(X) \leq (\geq)\ \varepsilon^w(X) = E(m^*_F(X)), \forall\alpha > 1\ (0 < \alpha < 1),$$

where the last equality follows by Proposition 2.1 of Misagh et al. (2011) and the proof is completed. □

Table 2 gives the expressions for $m^*_F(0)$ and $\varepsilon^w_\alpha(X)$ using Exponential (λ) and **Pareto** (θ,β) distributions and it is easy to verify the bounds (a) and (b).

3 Dynamic weighted survival entropy

Length of time during a study period has been considered as a prime variable of interest in many fields such as reliability, survival analysis, economics, business etc. In particular, consider an item under study, then the information about the residual lifetime is an important task in many applications. In such cases, the information **measures are functions of time, and thus they** are dynamic. In the following, we define the dynamic version of WES and study its important properties.

Definition 3.1. *The dynamic weighted survival entropy (DWSE) of order* α *of a random lifetime* X *is the weighted entropy for the random variable* $X|X > t$ *and is defined as*

$$\varepsilon^w_\alpha(X;t) = \frac{1}{1-\alpha}\log\int_t^\infty x\left(\frac{\bar{F}(x)}{\bar{F}(t)}\right)^\alpha dx, \quad \alpha > 0,\ \alpha \neq 1. \tag{14}$$

It is easy to verify that for each t≥ 0, $\varepsilon^w_\alpha(X;t)$ possess all the properties of $\varepsilon^w_\alpha(X)$. It is obvious that $\varepsilon_\alpha^w(X;0) = \varepsilon_\alpha^w(X)$.

In the following, we extend Lemma 2.1 and Theorem 2.1 for DWSE.

Lemma 3.1. *If Y=aX+b, with a>0 and b≥ 0, then*

$$\bar{F}^\alpha\left(\frac{t-b}{a}\right)\exp\left[(1-\alpha)\varepsilon_\alpha^w\left(Y;\frac{t-b}{a}\right)\right] = a^2\bar{F}^\alpha\left(\frac{t-b}{a}\right)\exp\left[(1-\alpha)\varepsilon_\alpha^w\left(X;\frac{t-b}{a}\right)\right] + ab\bar{F}^\alpha\left(\frac{t-b}{a}\right)\exp\left[(1-\alpha)\varepsilon_\alpha\left(X;\frac{t-b}{a}\right)\right]. \tag{15}$$

Corollary 3.1. *From Lemma 3.1, we have*

(i) If b=0 in (15), we get

$$\bar{F}^\alpha\left(\frac{t}{a}\right)\exp\left[(1-\alpha)\varepsilon_\alpha^w\left(Y;\frac{t}{a}\right)\right] = a^2\bar{F}^\alpha\left(\frac{t}{a}\right)\exp\left[(1-\alpha)\varepsilon_\alpha^w\left(X;\frac{t}{a}\right)\right].$$

(ii) If a=1 in (15), we get

$$\bar{F}^\alpha(t-b)\exp\left[(1-\alpha)\varepsilon_\alpha^w(Y;t-b)\right] = \bar{F}^\alpha(t-b)\exp\left[(1-\alpha)\varepsilon_\alpha^w(X;t-b)\right] + b\bar{F}^\alpha(t-b)\exp\left[(1-\alpha)\varepsilon_\alpha(X;t-b)\right].$$

Theorem 3.1. *Let X be a non-negative continuous random variable with WMRL function $m^*(t)$ and DWSE $\varepsilon_\alpha^w(X;t)$ such that $\varepsilon_\alpha^w(X;t) < \infty$, then*

(a) $\varepsilon_\alpha^w(X;t) \geq \frac{1}{1-\alpha}\log m_F^*(t)$, *for* $\alpha > 0$.

(b) $\varepsilon_\alpha^w(X;t) \leq (\geq)\ \varepsilon^w(X;t), \quad \forall\alpha > 1\ (0 < \alpha < 1)$.

Proof. The result follows in the manner of Theorem 2.1. □

Abasnejad et al. (2010) have been defined two non-parametric classes of distributions based on the monotonicity properties of DSE. In Definition 3.2, we introduced two non-parametric classes of distributions based on the monotonicity properties of DWSE.

Definition 3.2. *The distribution function F is said to be increasing (decreasing) in dynamic weighted survival entropy, IDWSE (DDWSEE), if* $\varepsilon^w_\alpha(X;t)$ *is an increasing (decreasing) function of t.*

The following counter example shows that there exist distributions which are not monotone in terms of $\varepsilon^w_\alpha(X;t)$.

Counterexample 3.1. *Let X be a random variable with probability density function* $f(x) = \begin{cases} \frac{1}{2}; & 0 \le X \le 2 \\ 0; & otherwise. \end{cases}$

Take $\alpha = 0.2$. Then, we see that for all $t \ge 0$, $\varepsilon^w_\alpha(X;t) = \frac{1}{1-\alpha}\log\left[\frac{(2-t)(2+t(\alpha+1))}{(\alpha+1)(\alpha+2)}\right]$, which is not monotone in $t \in [0, 0.4]$ as shown in Figure 1(a). Again for $\alpha = 5.2$, we see that for all $t \ge 0$, $\varepsilon^w_\alpha(X;t)$ is not monotone in $t \in [0, 2]$ as shown in figure 1(b). Hence $\varepsilon^w_\alpha(X;t)$ is not monotone.

The following theorem gives the upper (respective lower) bound to the DWSE, in terms of the hazard rate function $h(t) = -\frac{d}{dt}\log\bar{F}(t)$.

Theorem 3.2. *The distribution function F is IDWSE (DDWSE), if and only if, for all t*≥ 0

$$\varepsilon^w_\alpha(X;t) \ge (\le)\frac{\log\alpha}{\alpha-1} + \frac{1}{\alpha-1}\log\frac{h(t)}{t}, \quad \forall\alpha > 0.$$

Proof. From (14), we have

$$\bar{F}^\alpha(t)\exp\{(1-\alpha)\varepsilon^w_\alpha(X;t)\} = \int_t^\infty x\,\bar{F}^\alpha(x)\,dx. \tag{16}$$

Differentiating (16) with respect to t, we get

$$\frac{d}{dt}\varepsilon^w_\alpha(X;t) = \frac{1}{1-\alpha}\left[\alpha h(t) - t\exp\{(\alpha-1)\varepsilon^w_\alpha(X;t)\}\right], \tag{17}$$

and the result follows. □

Example 3.1: Let X have a Uniform distribution on (0, 2). It can be easily shown that $\varepsilon^w_\alpha(X;t)= \frac{1}{1-\alpha}\log(2-t) + \frac{1}{1-\alpha}\log(2+t(\alpha+1)) - \frac{1}{1-\alpha}\log[(\alpha+1)(\alpha+2)]$. Also $h(t) = \frac{1}{2-t}$. In Figure 2(a), we gives plots of $\varepsilon^w_\alpha(X;t)$, $h(t)$ for $t \in [0, 1.5]$. Here F is IDWSE. **Example 3.2:** Let X have Pareto distribution with probability density function as given in Table 3. It can be easily shown that

$\varepsilon_\alpha^w(X;t)=\frac{1}{1-\alpha}\log(t+\beta)+\frac{1}{1-\alpha}\log[t(\theta\alpha-1)+\beta]-\frac{1}{1-\alpha}\log[(\theta\alpha-1)(\theta\alpha-2)]$. Also, $h(t)=\theta(t+\beta)^{-1}$. In Figure 2(b), we gives plots of $\varepsilon_\alpha^w(X;t)$, $h(t)$ for $t\in[0,2]$. Here we can see that, F is DDWSE.

The following Theorem characterizes DWSE in the sense that it uniquely determine the underlying distribution.

Theorem 3.3. Let X and Y be two non-negative absolutely continuous random variable with survival functions $\bar{F}$ and $\bar{G}$ and the hazard functions $h_1(t)$ and $h_2(t)$ respectively. Let DWSE's $\varepsilon_\alpha^w(X;t)$ and $\varepsilon_\alpha^w(Y;t)$ corresponding to X and Y be increasing functions of t. If for all $t\geq 0$, $\varepsilon_\alpha^w(X;t)$=$\varepsilon_\alpha^w(Y;t)$, then, $\bar{F}(t)=\bar{G}(t)$.

Proof. Differentiating both sides of $\varepsilon_\alpha^w(X;t)$=$\varepsilon_\alpha^w(Y;t)$ with respect to t, and using (17), we get

$$\frac{1}{\alpha-1}\left\{te^{(\alpha-1)\varepsilon_\alpha^w(X;t)}-\alpha h_1(t)\right\}=\frac{1}{\alpha-1}\left\{te^{(\alpha-1)\varepsilon_\alpha^w(Y;t)}-\alpha h_2(t)\right\},$$

which implies that $h_1(t)=h_2(t)$ or equivalently $\bar{F}(t)=\bar{G}(t)$. □

4 Stochastic ordering of WSE and DWSE

In this section, we discuss the WSE ordering and the DWSE ordering of random variables. In the following definition, let X and Y denote random variables with distribution functions F and G, density functions f and g and survival functions $\bar{F}$ and $\bar{G}$.

Definition 4.1. *The random variable X is said to be less than or equal to Y in the*

(a) Likelihood ratio ordering, denoted by $X\overset{lr}{\leq}Y$, if $\frac{f(x)}{g(x)}$ is decreasing in x.

(b) Stochastic ordering, denoted by $X\overset{st}{\leq}Y$, if $\bar{F}(x)\leq\bar{G}(x)$, $\forall\, x\geq 0$.

(c) Hazard rate ordering, denoted by $X\overset{hr}{\leq}Y$, if $h_F(x)\geq h_G(x)$, $\forall\, x\geq 0$.

(d) Survival entropy ordering of order $\alpha>0$, denoted by $X\overset{se(\alpha)}{\leq}Y$, if $\varepsilon_\alpha(X)\leq\varepsilon_\alpha(Y)$.

(e) Weighted Survival entropy ordering of order $\alpha>0$, denoted by $X\overset{wse(\alpha)}{\leq}Y$, if $\varepsilon_\alpha^w(X)\leq\varepsilon_\alpha^w(Y)$.

The following theorem gives weighted survival entropy ordering.

Theorem 4.1. *Let X and Y be two non-negative absolutely continuous random variable with survival functions $\bar{F}$ and $\bar{G}$. If $X \overset{st}{\leq} Y$, then*

$$X \overset{wse(\alpha)}{\geq} (\overset{wse(\alpha)}{\leq}) Y, \ \ \forall \alpha > 1 \ (0 < \alpha < 1).$$

Table 4 shows some well-known families of distributions that can be ordered using $\varepsilon_\alpha^w(X)$. For some distributions such as Exponential or **Pareto**, $\varepsilon_\alpha^w(X)$ can be computed in closed form and thus, the ordering can be easily obtained. But for ordering other distributions we use $Theorem$ 4.1 and also likelihood ratio and stochastic ordering. For example, it can be easily shown that, if X has a gamma distribution with shape parameter θ, then for $\theta_0 < \theta_1$, $X_{\theta_0} \overset{lr}{\leq} X_{\theta_1}$, and thus $X_{\theta_0} \overset{st}{\leq} X_{\theta_1}$. So we have $X_{\theta_0} \overset{wse(\alpha)}{\geq} (\overset{wse(\alpha)}{\leq}) X_{\theta_1}$, $\forall \alpha > 1 \ (0 < \alpha < 1)$.

In the following theorem,we discuss the DWSE and hazard rate ordering.

Theorem 4.2. *Let X and Y be two non-negative absolutely continuous random variable with survival functions $\bar{F}$ and $\bar{G}$ and the hazard functions $h_F(t)$ and $h_G(t)$ respectively. If $X \overset{hr}{\geq} Y$, that is $h_F(t) \leq h_G(t)$ for all t≥ 0, then*

$$\varepsilon_\alpha^w(X;t) \leq (\geq) \ \varepsilon_\alpha^w(Y;t) \ \forall \alpha > 1 \ (0 < \alpha < 1)$$

In particular, $\varepsilon_\alpha^w(X) \leq (\geq) \ \varepsilon_\alpha^w(Y) \ \forall \alpha > 1 \ (0 < \alpha < 1)$.

The following example gives an application of Theorem 4.2 in the area of Order statistics.

Example 4.1: Let $X_1, X_2, ..., X_n$ be independent and identically distributed non-negative random variables having survival function $\bar{F}$. If $X_{i:n}$ denotes the i^{th} order statistic of $X_1, X_2, ..., X_n$, then we have following results.

(a) $\varepsilon_\alpha^w(X_{i:n};t) \geq (\leq) \ \varepsilon_\alpha^w(X_{i+1:n};t) \ \forall \alpha > 1 \ (0 < \alpha < 1)$. That is $\varepsilon_\alpha^w(X_{i:n};t)$ is a decreasing (increasing) function of i for all $\alpha > 1 \ (0 < \alpha < 1)$.

(b) $\varepsilon_\alpha^w(X_{1:n};t) \geq (\leq) \ \varepsilon_\alpha^w(X_{1:n-1};t) \ \forall \alpha > 1 \ (0 < \alpha < 1)$.

(c) $\varepsilon^w_\alpha(X_{n-1:n-1};t) \geq (\leq)\ \varepsilon^w_\alpha(X_{n:n};t)\ \forall\alpha > 1\ (0 < \alpha < 1)$.

5 Characterization of Rayleigh distribution

In the following Theorem, we show that the Rayleigh distribution can be characterized in terms of the DWSE.

Theorem 5.1. *Let X be an absolutely continuous random variable. Then the relation $\varepsilon^w_\alpha(X;t) = k$, where k is a constant holds if and only if X has the Rayleigh distribution with survival function*

$$\bar{F}(x) = \exp\left(\frac{-x^2}{2\sigma^2}\right). \tag{18}$$

Proof. The if part of the theorem can be easily obtained using (14). For the only if part, let the DWSE of X be a constant. Then $\frac{d}{dt}\varepsilon^w_\alpha(X;t) = 0$. Now using (17), we get

$$\frac{1}{1-\alpha}\left[\alpha h(t) - t\ \exp\{(\alpha-1)\varepsilon^w_\alpha(X;t)\}\right] = 0.$$

The above equation simplifies to h(t)= $\frac{t}{\sigma^2}$. Hence the proof is completed. □

In the following theorem, we characterize Rayleigh distribution using a relationship of DWSE with WMRL.

Theorem 5.2. *Let X be an absolutely continuous random variable. Then the relation*

$$(1-\alpha)\varepsilon^w_\alpha(X;t) = \log\left(\frac{m^*(t)}{\alpha}\right), \tag{19}$$

holds if and only if X has the survival function given in (18).

Proof. If X has a Rayleigh distribution with survival function given in (18), it can be easily shown that (19) holds with $m^*(t) = \sigma^2$. conversely, let (19) hold. Then using (14) and (13), we get

$$\bar{F}(t)\int_t^\infty x(\bar{F}(x))^\alpha dx = (\bar{F}(x))^\alpha \frac{1}{\alpha}\int_t^\infty x\bar{F}(x)dx.$$

Differentiating both sides of the above expression with respect to t, we get

$$-t\bar{F}(t)(\bar{F}(x))^{\alpha} - f(t)\int_t^{\infty} x(\bar{F}(x))^{\alpha}dx \quad = \quad -\frac{1}{\alpha}(\bar{F}(x))^{\alpha}t\bar{F}(t) - (\bar{F}(t))^{\alpha-1}f(t)\int_t^{\infty} x\bar{F}(x)dx.$$

The above equation simplifies to

$$h(t)m^*(t) = t, \quad \text{where } h(t) = \frac{f(t)}{\bar{F}(t)}. \tag{20}$$

Differentiating (13) with respect to t, we get

$$\frac{d}{dt}m^*(t)\bar{F}(t) - f(t)m^*(t) = -t\bar{F}(t).$$

Using (20), the above equation simplifies to $\frac{d}{dt}m^*(t) = 0$.

Or, equivalently,

$$m^*(t) = c, \quad \text{where c is a constant.} \tag{21}$$

Using (21) and (13), we get

$$h(t) = \frac{t}{c}.$$

Or

$$\bar{F}(t) = exp\left(\frac{-t^2}{c}\right),$$

which is the survival function of Rayleigh distribution and the result follows. □

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Table 1: Examples for Lemma 2.2

Density function	$\varepsilon_\alpha^w(X)$	$\varepsilon_\alpha^w(X\theta)$	$\varepsilon_\alpha^w(X_\theta^*)$
Uniform on (0,a) f(x)=$\frac{1}{a}$, $a > 0$	$\frac{1}{1-\alpha}\log[\frac{a^2}{(\alpha+1)(\alpha+2)}]$	$\frac{1}{1-\alpha}\log[\frac{a^2\theta^2}{(\alpha+1)(\alpha+2)}]$	$\frac{1}{1-\alpha}\log[\frac{a^2}{2+3\alpha\theta+\theta\alpha^2}]$
Exponential (λ) f(x)=$\lambda e^{-\lambda x}$, $\lambda > 0$	$\frac{2}{\alpha-1}(\log\lambda + \log\alpha)$	$\frac{2}{\alpha-1}(\log\lambda + \log\alpha - \log\theta)$	$\frac{2}{\alpha-1}(\log\lambda + \log\alpha + \log\theta)$

Table 2: Examples for Theorem 2.1

Density function	$m_F^*(0)$	$\varepsilon_\alpha^w(X)$
Exponential (λ) f(x)=$\lambda e^{-\lambda x}, \lambda > 0$	$\frac{1}{\lambda^2}$	$\frac{2}{\lambda^2}$
Pareto (θ,β) f(x)=$\theta\beta(x+\beta)^{-\theta-1}, \theta > 1, \beta > 0$.	$\frac{\beta^2}{(\theta-1)(\theta-2)}$	$\frac{\beta^2\theta(2\theta-3)}{[(\theta-1)(\theta-2)]^2}$

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Table 3: Examples for Theorem 3.1

Density function	$m_F^*(t)$	$\varepsilon_\alpha^w(X;t)$
Uniform $(0, \theta)$ f(x)=$\frac{1}{\theta}$, $\theta > 0$	$\frac{(\theta+2t)(\theta-t)}{6}$	$\frac{1}{1-\alpha}\log(\theta - t) + \frac{1}{1-\alpha}\log[\theta + t(\alpha+1)] - \frac{1}{1-\alpha}\log[(\alpha+1)(\alpha+2)]$
Exponential (λ) f(x)=$\lambda e^{-\lambda x}$, $\lambda > 0$	$\frac{\lambda t+1}{\lambda^2}$	$\frac{1}{1-\alpha}\log(\lambda\alpha t + 1) - \frac{2}{1-\alpha}\log(\lambda\alpha)$
Pareto (θ, β) f(x)=$\theta\beta(x+\beta)^{-\theta-1}$, $\theta > 1$, $\beta > 0$.	$\frac{(t+\beta)(\theta t - t+\beta)}{(\theta-1)(\theta-2)}$	$\frac{1}{1-\alpha}\log(t+\beta) + \frac{1}{1-\alpha}\log[t(\theta\alpha - 1) + \beta] - \frac{1}{1-\alpha}\log[(\theta\alpha-1)(\theta\alpha-2)]$

Table 4: **WSE ordering for some well-known distributions.**

Family and density	Ordering ($\alpha > 1$)	Ordering ($0 < \alpha < 1$)
Uniforn (0,θ) f(x)=$\frac{1}{\theta}, \theta > 0$	Decreasing	Increasing
Exponential [0,∞) f(x)=$\lambda e^{-\lambda x}, \lambda > 0$	Increasing	Decreasing
Pareto [0,∞) f(x)=$\theta\beta x^{\beta-1}e^{-\theta x^{\beta}}, \theta > 0, 0 < \beta < 1.$	Decreasing(in β) Decreasing(in θ)	Increasing(in β) Increasing(in θ)
Pareto [0,∞) f(x)=$\theta\beta(x+\beta)^{-\theta-1}, \theta > 1, \beta > 0.$	Decreasing(in β) Increasing(in θ)	Increasing(in β) Decreasing(in θ)

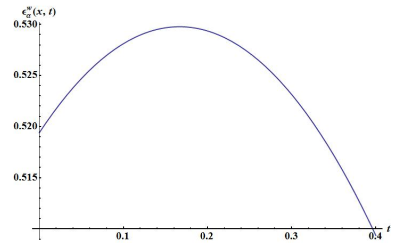

Figure 1(a): Plot of $\varepsilon_\alpha^w(X;t)$ for $t \in [0, 0.4]$ and $0 < \alpha < 1$ (Counter example 3.1)

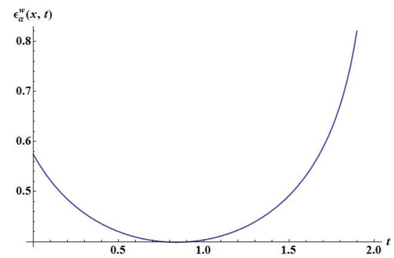

Figure 1(b): Plot of $\varepsilon_\alpha^w(X;t)$ for $t \in [0, 2]$ and $\alpha > 1$ (Counter example 3.1)

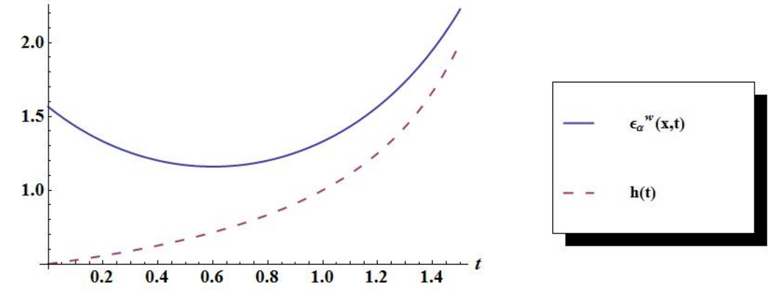

Figure 2(a): Plots of $\varepsilon_{\alpha}^{w}(X;t)$ and $h(t)$ for $0 \leq t \leq 1.5$ (Example 3.1)

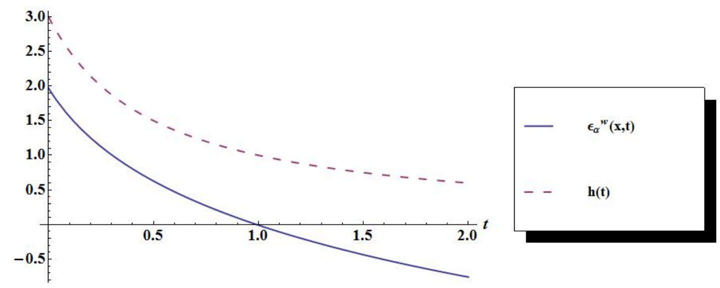

Figure 2(b): Plots of $\varepsilon_{\alpha}^{w}(X;t)$ and $h(t)$ for $0 \leq t \leq 2$ (Example 3.2)