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A Study on Dynamic Weighted Failure Entropy of Order α

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SYNOPTIC ABSTRACT

Abbasnejad (2011) proposed new measures of information known as failure and dynamic failure entropy of order α based on generalization of the well-known cumulative entropy. In this article, we propose the weighted forms of failure entropy (WFE) and dynamic failure entropy (DWFE) of order α. Some relationships of these measures with well-known reliability measures and ageing classes are studied and some characterization results for reflected Weibull distribution are provided. We also propose an empirical estimator for the WDFE of order α. We demonstrate the performance of the proposed estimator with a real life data set and further carried out a comprehensive Monte-Carlo simulation study.

KEY WORDS AND PHRASES

Life distributions; failure entropy; ordering; reliability

1. Introduction

A fundamental uncertainty measure of a random variable is known as entropy and was introduced by Shannon (1948). If X is a non-negative random variable having an absolutely continuous distribution function F with probability density function f, then the entropy of X is defined as follows.

$$H(X) = -\int_0^{+\infty} f(x) \log f(x)\, dx. \tag{1}$$

From Table 1, we can observe that the measure defined in Equation (1) can take values from $-\infty$ to $+\infty$. Shannon's entropy gives equal importance or weight to the occurrence of every event. Weighted entropy, which is a generalization of classical entropy, has been proposed by Belis and Guiasu (1968) and is defined as follows.

$$H^w(X) = -\int_0^{+\infty} x f(x) \log f(x)\, dx. \tag{2}$$

The factor x, in the integrand of Equation (2) represents a weight which linearly emphasizes the occurrence of the event $\{X = x\}$. This is a length biased shift dependent information measure assigning greater importance to larger values of X. From Table 1, we can observe that the measure defined in Equation (2) can take values from $-\infty$ to $+\infty$. Di Crescenzo and Longobardi (2006) introduced the notions of weighted residual entropy and weighted past entropy that are suitable to describe dynamic information of random life times. They pointed out the application of these measures in reliability, neurobiology, etc.

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Table 1. Values of $H(X)$, $H^w(X)$, $\varepsilon(X)$, $\bar{\varepsilon}(X)$, $\bar{\varepsilon}^w(X)$, and $\bar{\varepsilon}_\alpha(X)$ for Uniform $(0, a)$ with various values of a.

Entropy measure	$a = 1$	$0 < a < 1$ ($a = 0.5$)	$a > 1$ ($a = 4$)
Shannon Entropy $H(X)$	0	− 0.69314	1.38269
Weighted entropy $H^w(X)$	0	− 0.1733	2.7725
CRE $\varepsilon(X)$	0.25	0.125	1
CE $\bar{\varepsilon}(X)$	0.25	0.125	1
WCE $\bar{\varepsilon}^w(X)$	0.1111	0.125	1.7778
FE $\bar{\varepsilon}_\alpha(X)$ $(0 < \alpha < 1)$	− 0.8109	− 2.1972	1.9616
$(\alpha > 1)$	0.8351	1.2972	− 0.0802

An alternative to Shannon's entropy, namely cumulative residual entropy (CRE), has been defined by Rao et al. (2004) and is defined as follows.

$$\varepsilon(X) = -\int_0^\infty \bar{F}(x) \log \bar{F}(x)\, dx,$$

where $\bar{F}(x) = 1 - F(x)$ denotes the survival function of X. Note, that CRE is always non-negative and its definition is valid in the continuous and discrete domain. From Table 1, we can observe that $\varepsilon(X)$ is always non-negative.

Asadi and Zohrevand (2007) introduced a dynamic version of CRE called dynamic cumulative residual entropy (DCRE). Di Crescenzo and Longobardi (2009) introduced a new information measure similar to $\varepsilon(X)$, namely cumulative entropy (CE) based on the distribution function $F(x)$ of X, and is given by the following.

$$\bar{\varepsilon}(X) = -\int_0^\infty F(x) \log F(x)\, dx. \tag{3}$$

Note that CE is always non-negative. From Table 1, we can observe that $\bar{\varepsilon}(X)$ is always non-negative.

Zografos and Nadarajah (2005) introduced a measure of uncertainty, called survival exponential entropy. Abasnejad (2010) proposed a measure of uncertainty based on the survival function, called survival entropy of order α.

Misagh et al. (2011) proposed a weighted information measure that is based on CE called weighted cumulative entropy (WCE), defined by the following.

$$\bar{\varepsilon}^w(X) = -\int_0^\infty x F(x) \log F(x)\, dx. \tag{4}$$

Note that WCE is always non-negative. From Table 1, we can observe that $\bar{\varepsilon}^w(X)$ is always non-negative.

Similarly to Abasnejad (2010), Rajesh et al. (2017) proposed a measure of uncertainty called the weighted survival entropy of order α (WSE). Similar to Di Crescenzo and Longobardi (2009) and Zografos and Nadarajah (2005), Abasnejad (2011) proposed a measure of uncertainty based on the distribution function called the failure entropy of order α (FE), and is defined as follows.

$$\bar{\varepsilon}_\alpha(X) = -\frac{1}{\alpha - 1} \log \int_0^\infty F^\alpha(x)\, dx \ \forall \alpha > 0 \ (\alpha \neq 1). \tag{5}$$

From Table 1, we can observe that that the measure defined in Equation (5) can take values from $-\infty$ to $+\infty$. In Table 1, we present the possible values of some important entropy measures for Uniform distribution with probability density function $f(x) = \frac{1}{a}$; $0 \leq x \leq a$.

In this article, we derive the weighted version of FE defined in Equation (5) and it is termed weighted failure entropy of order α (WFE). We study various properties of this measure.

The rest of the article is organized as follows. In section 2, we propose WFE of order α, and properties including linear transformation and bounds for WFE of order α using reliability measures are discussed. In section 3, we propose dynamic weighted failure entropy (DWFE) of order α and obtained some bounds for it based on weighted mean inactivity time (WMIT). section 4 gives stochastic ordering of WFE and DWFE. Section 5 gives some characterization results for reflected Weibull distribution based on DWFE. Finally in Section 6, the empirical DWFE is proposed. A Monte Carlo simulation study is carried out to study the behavior of the estimator. The method is illustrated using a real data set.

2. Weighted Failure Entropy Properties

In this section, we introduce the weighted version of FE of order α and study some of its properties, including linear transformation and bounds for WFE using some reliability measures.

Definition 1. Let X be a non-negative random variable having an absolutely continuous distribution function F. The WFE order α of X is defined as

$$\bar{\varepsilon}_{\alpha}^{w}(X) = \frac{-1}{\alpha - 1} \log \int_{0}^{\infty} xF^{\alpha}(x)dx \quad \forall \alpha > 0\,(\alpha \neq 1). \tag{6}$$

It is easy to observe that the measure defined in (6) can take values from $-\infty$ to $+\infty$. For example, consider a random variable X with distribution function $F(x) = \frac{x}{a}$. If $a = 1$, $\bar{\varepsilon}_{\alpha}^{w}(X) = -1.8326$ for $\alpha = 0.5(0 < \alpha < 1)$, 1.0027 for $\alpha = 2.5(\alpha > 1)$. If $a = 0.5$ $(0 < a < 1)$, $\bar{\varepsilon}_{\alpha}^{w}(X) = -4.6051$ for $\alpha = 0.5(0 < \alpha < 1)$, 1.9269 for $\alpha = 2.5(\alpha > 1)$. Again if $a = 4$ $(a > 1)$, $\bar{\varepsilon}_{\alpha}^{w}(X) = 3.7126$ for $\alpha = 0.5(0 < \alpha < 1)$, -0.8456 for $\alpha = 2.5(\alpha > 1)$. Therefore, it has to be assumed that $\bar{\varepsilon}_{\alpha}^{w}(X)$ can take values from $-\infty$ to $+\infty$.

The following example shows that even if two distributions have the same FE, they can have different WFE.

Example 1. Let X and Y be random variables with respective density functions,

$$f_X(x) = \begin{cases} \frac{1}{2}, & 2 \leq X \leq 4 \\ 0, & otherwise \end{cases}$$

and

$$f_Y(y) = \begin{cases} \frac{1}{2}, & 4 \leq Y \leq 6 \\ 0, & otherwise. \end{cases}$$

It can be shown that,

$$\bar{\varepsilon}_{\alpha}(X) = \bar{\varepsilon}_{\alpha}(Y) = \frac{1}{1-\alpha} \log\left(\frac{2}{1+\alpha}\right),$$

where $\bar{\varepsilon}_{\alpha}(X)$ and $\bar{\varepsilon}_{\alpha}(Y)$ are the FE of X and Y, respectively. In particular, if $\alpha = 0.5$, $\bar{\varepsilon}_{\alpha}(X) = \bar{\varepsilon}_{\alpha}(Y) = 0.5754$ and if $\alpha = 5$, then $\bar{\varepsilon}_{\alpha}(X) = \bar{\varepsilon}_{\alpha}(Y) = 0.2746$. Therefore, the expected uncertainties in the predictability of failure time of component of X and Y are identical.

Now, the WFE of order α of the random variables X and Y are simplified as follows.

$$\bar{\varepsilon}_{\alpha}^{w}(X) = \frac{1}{1-\alpha} \log\left(\frac{12 + 8\alpha}{(\alpha+1)(\alpha+2)}\right)$$

and

$$\bar{\varepsilon}^w_\alpha(Y) = \frac{1}{1-\alpha}\log\left(\frac{4(5+3\alpha)}{(\alpha+1)(\alpha+2)}\right)$$

respectively. That is, $\bar{\varepsilon}^w_\alpha(X) \neq \bar{\varepsilon}^w_\alpha(Y)$. In particular, if $\alpha = 0.5$, then $\bar{\varepsilon}^w_\alpha(X) = 2.9017$ and $\bar{\varepsilon}^w_\alpha(Y) = 3.8726$, and if $\alpha = 5$, then $\bar{\varepsilon}^w_\alpha(X) = -0.0534$ and $\bar{\varepsilon}^w_\alpha(Y) = -0.1611$. Hence, even though $\bar{\varepsilon}_\alpha(X) = \bar{\varepsilon}_\alpha(Y)$, when $\alpha > 1 (0 < \alpha < 1)$, the expected weighted uncertainty of the predictability of the failure time of component X is larger (smaller) than that of Y.

In the following lemma, we discuss the effect of linear transformation of $\bar{\varepsilon}^w_\alpha(X)$.

Lemma 1. *If Y=aX+b, with a>0 and b≥ 0, then*

$$\exp\{(1-\alpha)\bar{\varepsilon}^w_\alpha(Y)\} = a^2\exp\{(1-\alpha)\bar{\varepsilon}^w_\alpha(X)\} + ab\exp\{(1-\alpha)\bar{\varepsilon}_\alpha(X)\}. \tag{7}$$

Proof. The proof follows by using the property of distribution function given by, $F_{aX+b}(x) = F_X(\frac{x-b}{a})$.

If $F_\theta{}^*(x)$ and $F(x)$ denote the distribution functions of the random variables X^*_θ and X, respectively, then the proportional reversed hazard model is described by the relation $F_\theta{}^*(x) = [F(x)]^\theta$, where θ is a real number.

In Lemma 2, we compare the $\bar{\varepsilon}^w_\alpha(X)$ of X, X^*_θ and θX. The proof is omitted.

Lemma 2. *For the WFE of order α, the following properties hold:*

(a) $\bar{\varepsilon}^w_\alpha(X^*_\theta) = \frac{\theta\alpha-1}{\alpha-1}\bar{\varepsilon}^w_{\alpha\theta}(X)$.
(b) $\bar{\varepsilon}^w_\alpha(X^*_\theta) \geq \bar{\varepsilon}^w_\alpha(X) \geq \bar{\varepsilon}^w_\alpha(X\theta)$, *for* $0 < \theta \leq 1, 0 < \alpha < 1$ *and* $\theta \geq 1, \alpha > 1$.
(c) $\bar{\varepsilon}^w_\alpha(X^*_\theta) \leq \bar{\varepsilon}^w_\alpha(X) \leq \bar{\varepsilon}^w_\alpha(X\theta)$, *for* $0 < \theta \leq 1, \alpha > 1$ *and* $\theta \geq 1, 0 < \alpha < 1$.

We present some distributions in Table 2 to verify the results $(a) - (c)$ stated in Lemma 2.

Let X be a continuous non-negative random variable with distribution function F. The weighted mean inactivity time (WMIT) of X is defined as follows.

$$\mu^*(t) = \frac{1}{F(t)}\int_0^t xF(x)dx. \tag{8}$$

In the following theorem, we give some bounds for the WFE using WMIT.

Theorem 1. *Let X be a non-negative continuous random variable with WMIT $\mu^*(t)$, WFE $\bar{\varepsilon}^w_\alpha(X)$ and WCE $c\varepsilon^w(X)$. Then*

(a) $\bar{\varepsilon}^w_\alpha(X) \geq \frac{1}{1-\alpha}\log\int_0^\infty xF(x)\,dx$, $\forall\alpha > 0$.
(b) $\bar{\varepsilon}^w_\alpha(X) \leq (\geq) E(\mu^*(X))$, $\forall\alpha > 1\ (0 < \alpha < 1)$.

Proof.

(a) Since $xF^\alpha(x) \leq (\geq)\, xF(x)$, for $\alpha > 1\ (0 < \alpha < 1)$ and integrating both sides of this inequality with respect to x, the result follows.

Table 2. Expressions of $\bar{\varepsilon}^w_\alpha(X)$, $\bar{\varepsilon}^w_\alpha(X\theta)$ and $\bar{\varepsilon}^w_\alpha(X^*_\theta)$ for some well-known distributions.

Density function	$\bar{\varepsilon}^w_\alpha(X)$	$\bar{\varepsilon}^w_\alpha(X\theta)$	$\bar{\varepsilon}^w_\alpha(X^*_\theta)$
Uniform on $(0, a)$ $f(x) = \frac{1}{a}, a > 0.$	$\frac{1}{1-\alpha}\log\left[\frac{a^2}{(\alpha+2)}\right]$	$\frac{1}{1-\alpha}\log\left[\frac{a^2\theta^2}{(\alpha+2)}\right]$	$\frac{1}{1-\alpha}\log\left[\frac{a^2}{2+\alpha\theta}\right]$
Type 3 extreme value $f(x) = e^{c(x-b)}, x < b, c > 0.$	$\frac{1}{1-\alpha}\log\left[\frac{bc\alpha-1}{(c\alpha)^2}\right]$	$\frac{1}{1-\alpha}\log\left[\frac{(bc\alpha-1)\theta^2}{(c\alpha)^2}\right]$	$\frac{1}{1-\alpha}\log\left[\frac{bc\alpha\theta-1}{(c\alpha\theta)^2}\right]$

(b) It can be shown that

$$\bar{\varepsilon}^{w}_{\alpha}(X) \le (\ge)\, c\varepsilon^{w}(X) = E(\mu^{*}(X)),\, \forall\alpha > 1\, (0 < \alpha < 1),$$

where the last equality follows by Proposition 2.1 of Misagh et al. (2011).

3. Dynamic Weighted Failure Entropy

Length of time during a study period has been considered as a prime variable of interest in many fields such as reliability, survival analysis, economics, business, etc. In particular, if we consider an item under study, then the information about the past lifetime is an important task in many applications. For instance, consider a system whose state is observed only at certain preassigned inspection times. If at time t the system is inspected for the first time and it is found to be down, then the uncertainty relies on the past time, i.e., on which instant in $(0, t)$ it has failed. In such cases, the information measures are functions of time, and thus, they are dynamic. In the following, we define the dynamic version of WFE which then computes the uncertainty related to the past.

Definition 2. The dynamic weighted failure entropy (DWFE) of order α is the weighted failure entropy for the random variable $X_t = [t - X|X \le t]$ and is defined as

$$\bar{\varepsilon}^{w}_{\alpha}(X; t) = \frac{-1}{\alpha - 1} \log \int_{a}^{t} x \left(\frac{F(x)}{F(t)}\right)^{\alpha} dx, \quad \alpha > 0,\ \alpha \ne 1. \tag{9}$$

It is easy to verify that for each $t \ge 0$, $\bar{\varepsilon}^{w}_{\alpha}(X; t)$ possess all the properties of $\bar{\varepsilon}^{w}_{\alpha}(X)$. It is then obvious that $\bar{\varepsilon}^{w}_{\alpha}(X; 0) = \bar{\varepsilon}^{w}_{\alpha}(X)$.

In the following lemma shows effect of linear transformation of $\bar{\varepsilon}^{w}_{\alpha}(X; t)$.

Lemma 3. *If $Y = aX + b$, with $a > 0$ and $b \ge 0$, then*

$$\begin{aligned} F^{\alpha}\left(\frac{t-b}{a}\right) \exp\left[(1-\alpha)\bar{\varepsilon}^{w}_{\alpha}\left(Y; \frac{t-b}{a}\right)\right] &= \exp\left[(1-\alpha)\bar{\varepsilon}^{w}_{\alpha}\left(X; \frac{t-b}{a}\right)\right] \\ &\quad \times a^{2}F^{\alpha}\left(\frac{t-b}{a}\right) + ab\,F^{\alpha}\left(\frac{t-b}{a}\right) \\ &\quad \times \exp\left[(1-\alpha)\bar{\varepsilon}_{\alpha}\left(X; \frac{t-b}{a}\right)\right]. \end{aligned} \tag{10}$$

In the following theorem, we give some bounds for $\bar{\varepsilon}^{w}_{\alpha}(X; t)$.

Theorem 2. *Let X be a non-negative continuous random variable with WMIT $\mu^{*}(t)$ and DWFE $\bar{\varepsilon}^{w}_{\alpha}(X; t)$ such that $\bar{\varepsilon}^{w}_{\alpha}(X; t) < \infty$. Then*

(a) $\bar{\varepsilon}^{w}_{\alpha}(X; t) \ge \frac{-1}{\alpha-1} \log \mu^{*}(t)$, for $\alpha > 0$.
(b) $\bar{\varepsilon}^{w}_{\alpha}(X; t) \le (\ge)\, c\varepsilon^{w}(X; t)$, for $\alpha > 1\ (0 < \alpha < 1)$.

Proof.
(a) The result follows similar to the proof of (a) in Theorem 1.
(b) The proof is similar to Theorem 5.2 of Di Crescenzo and Longobardi (2009).

We present some well-known distributions in Table 3 to verify the results (a) and (b) stated in Theorem 2.

Table 3. Expressions of WMIT and DWFE for some well-known distributions.

Density function	$\mu^*(t)$	$\bar{\varepsilon}_\alpha^w(X;t)$
Uniform $(0,\theta)$ $F(x)=\frac{x}{\theta},\ \theta>0$	$\frac{t^2}{3}$	$\frac{1}{1-\alpha}\log\left[\frac{t^2}{(\alpha+2)}\right]$
Power distribution $F(x)=(\frac{x}{\beta})^k,\ \beta,k>0$	$\frac{t^2}{3}$	$\frac{1}{1-\alpha}\log\left[\frac{t^2}{(k\alpha+2)}\right]$
Type 3 extreme value $F(x)=e^{c(x-b)},\ c>0, x<b.$	$\frac{(ct-1)}{c^2}$	$\frac{1}{1-\alpha}\log\left[\frac{ct\alpha-1}{(c\alpha)^2}\right]$

Analogous to Abasnejad (2011), we now define two non-parametric classes of distributions based on the monotonicity properties of DWFE.

Definition 3. The distribution function F is said to be increasing (decreasing) in dynamic weighted failure entropy IDWFE (DDWFE), if $\bar{\varepsilon}_\alpha^w(X;t)$ is an increasing (decreasing) function of t.

The following counterexample shows that there exist distributions which are non-monotone in terms of $\bar{\varepsilon}_\alpha^w(X;t)$.

Counterexample 1. Let X be a random variable having distribution function

$$F(x)=\exp\left(1-\frac{1}{x}\right),\ 0\le x\le 1.$$

Then, for all $t\ge 0$,

$$\bar{\varepsilon}_\alpha^w(X;t)=\frac{1}{1-\alpha}\log\left[\frac{1}{2}\left(t(t-\alpha)-e^{(\frac{\alpha}{t})}\alpha^2\left(CoshIntegral\left[\frac{\alpha}{t}\right]\right)\right]\right.$$
$$\left.-\frac{1}{1-\alpha}\log\left[\frac{1}{2}\left(SinhIntegral\left[\frac{\alpha}{t}\right]\right)\right)\right],$$

where $CoshIntegral[.]$ and $SinhIntegral[.]$ gives the hyperbolic cosine and sine integrals, respectively. Clearly for $\alpha=1.5$, $\bar{\varepsilon}_\alpha^w(X;t)$ is not monotone in $t\in[0,2]$, as shown in Figure 1a. Again for $\alpha=0.5$, we see that for all $t\ge 0$, $\bar{\varepsilon}_\alpha^w(X;t)$ is not monotone in $t\in[0,2]$, as shown in Figure 1b. Hence, $\bar{\varepsilon}_\alpha^w(X;t)$ is non- monotone.

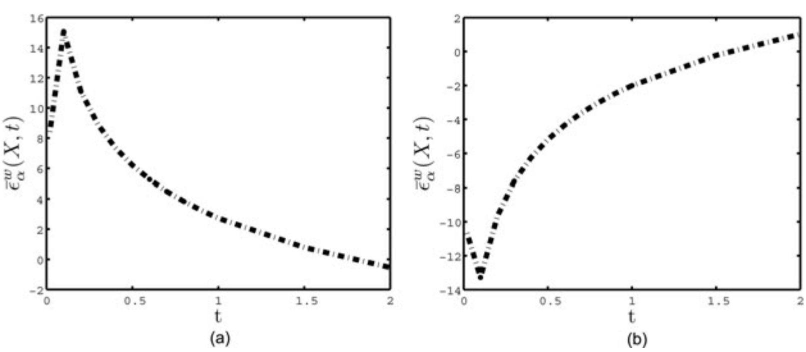

Figure 1. (a) Plot of $\bar{\varepsilon}_\alpha^w(X;t)$ for $t\in[0,2]$ and $\alpha>1$ (Counterexample 1). (b) Plot of $\bar{\varepsilon}_\alpha^w(X;t)$ for $t\in[0,2]$ and $0<\alpha<1$ (Counterexample 1).

Let X be a non-negative continuous random variable with distribution function $F(t)$, then the reversed hazard rate (RHR) function is defined as follows.

$$\phi(t) = \frac{d}{dt} \log F(t)$$

The following theorem gives the upper (respective lower) bound to the DWFE, in terms of the RHR function $\phi(t)$.

Theorem 3. *The distribution function F is IDWFE (DDWFE), if and only if, for all $t \geq 0$*

$$\bar{\varepsilon}_\alpha^w(X; t) \geq (\leq) \frac{\log \alpha}{\alpha - 1} + \frac{1}{\alpha - 1} \log \frac{\phi(t)}{t}, \quad \text{for } \alpha > 0.$$

Proof. From (9), we have,

$$F^\alpha(t) \exp\{(1 - \alpha)\bar{\varepsilon}_\alpha^w(X; t)\} = \int_0^t x F^\alpha(x)\, dx. \tag{11}$$

Differentiating Equation (11) with respect to t, and after some calculations, we obtain,

$$\frac{d}{dt}\bar{\varepsilon}_\alpha^w(X; t) = \frac{1}{\alpha - 1} \left\{t \exp\left[(\alpha - 1)\bar{\varepsilon}_\alpha^w(X; t)\right] - \alpha\phi(t)\right\} \tag{12}$$

and the result follows.

The following theorem proves that DWFE uniquely determines the underlying distribution.

Theorem 4. *Let X and Y be two non-negative absolutely continuous random variables with distribution functions F and G respectively and with reversed hazard functions $\phi_1(t)$ and $\phi_2(t)$, respectively. Let DWFE's $\bar{\varepsilon}_\alpha^w(X; t)$ and $\bar{\varepsilon}_\alpha^w(Y; t)$ defined corresponding to X and Y are increasing functions of t. If for all $t \geq 0$, $\bar{\varepsilon}_\alpha^w(X; t) = \bar{\varepsilon}_\alpha^w(Y; t)$, then, $F(t) = G(t)$.*

Proof. Differentiating both sides of $\bar{\varepsilon}_\alpha^w(X; t) = \bar{\varepsilon}_\alpha^w(Y; t)$ with respect to t and using (12), we get the following.

$$\frac{1}{\alpha - 1} \left\{t \exp\left[(\alpha - 1)\bar{\varepsilon}_\alpha^w(X; t)\right] - \alpha\phi_1(t)\right\} = \frac{1}{\alpha - 1} \left\{t \exp\left[(\alpha - 1)\bar{\varepsilon}_\alpha^w(X; t)\right] - \alpha\phi_2(t)\right\},$$

which implies that $\phi_1(t) = \phi_2(t)$ or equivalently $F(t) = G(t)$.

4. Stochastic Ordering of WFE and DWFE

In this section, we study the WFE ordering and DWFE ordering of two random variables. In the following definition, consider two random variables X and Y with distribution functions F and G and probability density functions $f(x)$ and $g(x)$, respectively.

Definition 4. The random variable X is said to be less than or equal to Y in the

(a) Likelihood ratio ordering, denoted by $X \overset{lr}{\leq} Y$, if $\frac{f(x)}{g(x)}$ is decreasing in x, $\forall x \geq 0$.

(b) Stochastic ordering, denoted by $X \overset{st}{\leq} Y$, if $\bar{F}(x) \leq \bar{G}(x)$, $\forall$ x ≥ 0.

(c) Reversed hazard rate ordering, denoted by $X \overset{rh}{\leq} Y$, if $\phi_F(x) \leq \phi(x)$, $\forall$ x ≥ 0.

(d) Failure entropy ordering of order $\alpha > 0$, denoted by $X \overset{fe(\alpha)}{\leq} Y$, if $\bar{\varepsilon}_\alpha(X) \leq \bar{\varepsilon}_\alpha(Y)$.

(e) Weighted failure entropy ordering of order $\alpha > 0$, denoted by $X \overset{wfe(\alpha)}{\leq} Y$, if $\bar{\varepsilon}_\alpha^w(X) \leq \bar{\varepsilon}_\alpha^w(Y)$.

Table 4. WFE ordering for some well-known distributions.

Family and density	Ordering ($\alpha > 1$)	Ordering ($0 < \alpha < 1$)
Uniform $(0, \theta)$	Increasing	Decreasing
$F(x) = \frac{x}{\theta}, \theta > 0$		
Power distribution	Increasing (in k)	Decreasing (in k)
$F(x) = (\frac{x}{\beta})^k,\ \beta, k > 0$	Decreasing (in β)	Increasing (in β)
Type 3 extreme value	Decreasing (in b)	Increasing (in b)
$F(x) = exp\{c(x - b)\},\ c > 0.$	Increasing (in c)	Decreasing (in c)

The following theorem gives implication with weighted failure entropy ordering and Stochastic ordering.

Theorem 5. *Let X and Y be two non-negative absolutely continuous random variables with distribution functions F and G, respectively. If $X \overset{st}{\leq} Y$, then*

$$X \overset{wfe(\alpha)}{\geq} \left(\overset{wfe(\alpha)}{\leq} \right) Y, \ for\, \alpha > 1\ (0 < \alpha < 1).$$

Table 4 shows some well-known families of distributions that can be ordered using $\bar{\varepsilon}^w_\alpha(X)$. In the following theorem, we discuss the DWFE and reversed hazard rate orderings.

Theorem 6. *Let X and Y be two non-negative absolutely continuous random variables with distribution functions F and G respectively and with reversed hazard functions $\phi_F(t)$ and $\phi_G(t)$, respectively. If $X \overset{rh}{\leq} Y$, that is $\phi_F(t) \leq \phi_G(t)$ for all $t \geq 0$, then,*

$$\bar{\varepsilon}^w_\alpha(X; t) \leq (\geq)\, \bar{\varepsilon}^w_\alpha(Y; t)\ for\, \alpha > 1\ (0 < \alpha < 1).$$

Proof. The result follows immediately using the fact that $F(t)G(x) \leq F(x)G(t)$ for all $x \leq t$.

The following example gives an application of Theorem 6 in the area of order statistics.

Example 2. The relation between the reversed hazard rate functions of $X_{n:n} = max(X_1, X_2, \ldots, X_n)$ and X is given by $\phi_{X_{n:n}}(t) = n\phi_X(t), t > 0$. So, $\phi_X(t) \leq \phi_{X_{n:n}}(t)$ or $X \overset{rh}{\leq} X_{n:n}$, then, $\bar{\varepsilon}^w_\alpha(X; t) \leq (\geq)\, \bar{\varepsilon}^w_\alpha(Y; t)$ for $\alpha > 1\ (0 < \alpha < 1)$.

5. Characterization Based on DWFE

In this section, we study some characterization results for reflected Weibull distribution (Lai and Xie, 2005). In the following theorem, we show that the reflected Weibull distribution can be characterized in terms of the DWFE.

Theorem 7. *Let X be an absolutely continuous random variable. Then, the relation $\bar{\varepsilon}^w_\alpha(X; t) = k$, where k is a constant holds if and only if X has the reflected Weibull distribution with distribution function.*

$$F(x) = e^{-bx^2}, -\infty < x < 0. \tag{13}$$

Proof. The if part of the theorem can be easily obtained using Equation (9). For the only if part, let the DWFE of X be a constant. Then, $\frac{d}{dt}\bar{\varepsilon}^w_\alpha(X; t) = 0$. Now using Equation (12), we get,

$$\frac{1}{1-\alpha}\left[\alpha\phi(t) - t\ \exp\{(\alpha-1)\bar{\varepsilon}_\alpha^w(X;t)\}\right] = 0.$$

The above equation simplifies to $\phi(t) = -kt$ or $F(t) = e^{-kt^2}$, where k is a constant.

In the following theorem, we characterize reflected Weibull distribution defined in Equation (13) using a relationship of DWFE with WMIT defined in Equation (8).

Theorem 8. *Let X be an absolutely continuous random variable. Then the relation,*

$$(1-\alpha)\bar{\varepsilon}_\alpha^w(X;t) = \log\left(\frac{\mu^*(t)}{\alpha}\right), \tag{14}$$

holds if and only if X has the distribution function given in Equation (13).

Proof. If X has a reflected Weibull distribution with distribution function given in Equation (13), then it can be easily shown that Equation (14) holds with $\mu^*(t) = -\frac{1}{2b}$. Conversely, suppose that Equation (14) holds. Then using Equations (9) and (8), respectively, we get,

$$F(t)\int_{-\infty}^{t} x(F(x))^\alpha dx = (F(t))^\alpha \frac{1}{\alpha}\int_{-\infty}^{t} xF(x)dx.$$

Differentiating both sides of the above expression with respect to t, we get,

$$tF(t)(F(x))^\alpha + f(t)\int_{-\infty}^{t} x(F(x))^\alpha dx = \frac{1}{\alpha}(F(x))^\alpha tF(t) + (F(t))^{\alpha-1}f(t)\int_{-\infty}^{t} xF(x)dx.$$

The above equation simplifies to the following.

$$\phi(t)\mu^*(t) = t, \ \text{where}\, \phi(t) = \frac{f(t)}{F(t)}. \tag{15}$$

Differentiating Equation (8) with respect to t, we get the following.

$$\frac{d}{dt}\mu^*(t)F(t) + f(t)\mu^*(t) = tF(t).$$

Using Equation (15), the above equation simplifies to $\frac{d}{dt}\mu^*(t) = 0$. Or, equivalently,

$$\mu^*(t) = pI(t < 0), \tag{16}$$

where I is an indicator function and p is a constant. Using Equations (16) and (8), respectively, we get

$$\phi(t) = -tp.$$

Consequently, we have $F(t) = e^{-pt^2}$, the distribution function of reflected Weibull distribution and, thus, the result follows.

In the following theorem, we characterize a finite range distribution using the relationship with DWFE and WMIT.

Theorem 9. *Let X be a random variable with distribution function F and WMIT $\mu^*(x)$. Then for a constant $c > 0$,*

$$\bar{\varepsilon}^{w}_{\alpha}(X;t) = c - \frac{1}{\alpha-1}\log\mu^*(t), \tag{17}$$

holds if and only if X follows finite range distribution with distribution function

$$F(x) = t^{\lambda},\ \ 0 < t < 1,\ \lambda > 0. \tag{18}$$

Proof. The WMIT for (18) simplifies to,

$$\mu^*(t) = \frac{t^2}{2+\beta}.$$

Now, representing c as

$$c = \frac{1}{1-\alpha}\log\frac{2+\beta}{2+\beta\alpha},$$

we get the if part of the theorem. For the only if part of the theorem, assume that Equation (17) holds, so that we have the following.

$$\int_0^t xF^{\alpha}(x)\,dx = \theta\mu^*(t)F^{\alpha}(t), \tag{19}$$

where $\theta = e^{-c(\alpha-1)}$. Differentiating both sides of Equation (19) with respect to t and using Equation (8), we get the following.

$$\frac{d}{dt}\mu^*(t) = at, \tag{20}$$

where

$$a = \frac{1-\alpha\theta}{\theta(1-\alpha)}.$$

Integrating both sides of Equation (20), we get

$$\int_0^t xF(x)\,dx = at^2F(t).$$

Now, differentiating the above equation with respect to t, we get $\phi(t) = \frac{\beta}{t}$, where β is a constant. It follows that $F(t) = t^{\beta}$, is the distribution function of a finite range distribution and, thus, the result follows.

6. Estimation of DWFE

In this section, we will construct an empirical estimator for DWFE defined in Equation (9). An empirical distribution function is the distribution function associated with the empirical measure of a sample. The empirical distribution function estimates the cumulative distribution function underlying of the points in the sample and converges with probability 1. The definition for empirical estimator for DWFE is as follows.

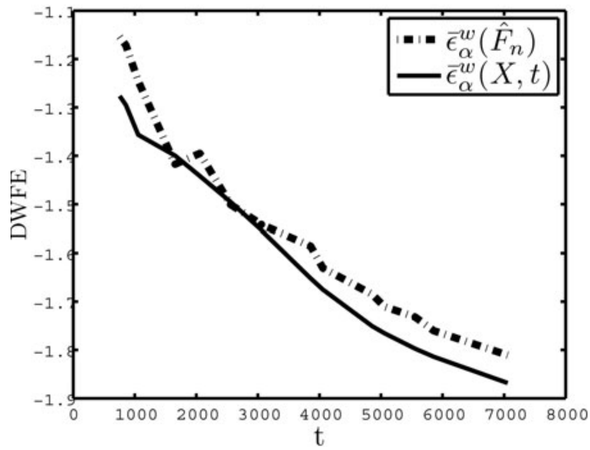

Figure 2. Plots of $\bar{\varepsilon}_{\alpha}^{w}(\hat{F}_n)$ and $\bar{\varepsilon}_{\alpha}^{w}(X;t)$ for the annual maxima stream flow of the Hillsborough River, Florida.

Definition 5. Let $X_1, X_2, \ldots, X_n$ be a random sample drawn from a population having distribution function F. From Equation (9), we define the empirical DWFE as follows.

$$\bar{\varepsilon}_{\alpha}^{w}(\hat{F}_n) = \frac{-1}{\alpha-1}\log\int_a^t x\left(\frac{\hat{F}_n(x)}{\hat{F}_n(t)}\right)^{\alpha}dx, \quad \alpha>0,\ \alpha\neq 1, \tag{21}$$

where $\hat{F}_n(x) = \frac{1}{n}\sum_{i=1}^{n} I\{X_i \le x\}$ is the empirical distribution and,

$$I\{X \le x\} = \begin{cases} 1, & X \le x \\ 0, & otherwise. \end{cases}$$

is the indicator function of the event $\{X \le x\}$.

6.1. Numerical Illustration

In this section, we illustrate the usefulness of the proposed estimator given in Equation (21) with real life situations.

Example 3. Consider the data-set reported by Miladinovic (2008). The data contains the annual maxima stream flow of the Hillsborough River, Florida, dating from 1940 to 2006.

Table 5. Bootstrap bias and MSE estimates of $\bar{\varepsilon}_{\alpha}^{w}(\hat{F}_n)$ for the annual maxima stream flow of the Hillsborough River, Florida.

	Bias and MSE of $\bar{\varepsilon}_{\alpha}^{w}(\hat{F}_n)$	
t	Bias	MSE
630	0.010	0.0014
670	− 0.0058	0.0008
960	− 0.0052	0.0001
1670	− 0.0328	0.0023
2670	− 0.0314	0.0011
5670	0.0626	0.0044
7670	0.0493	0.0025
8670	0.0403	0.0017
10670	0.0235	0.0006
11670	0.0182	0.0003

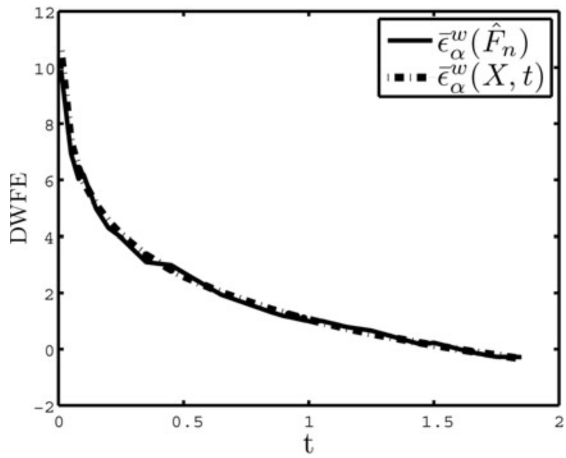

Figure 3. Plots of $\bar{\varepsilon}^w_\alpha(\hat{F}_n)$ and $\bar{\varepsilon}^w_\alpha(X;t)$ using a simulated sample of size=100, exponential (2), and $\alpha = 2$.

Here, we use the bootstrapping procedure. At each value of t, we calculate the bias and the mean-squared error of $\bar{\varepsilon}^w_\alpha(X;t)$ using 100 bootstrap samples of size 67. Figure 2 shows the plots of $\bar{\varepsilon}^w_\alpha(X;t)$ and $\bar{\varepsilon}^w_\alpha(\hat{F}_n)$ using the given data set for varoius values of t. From Figure 2, it is easy to see that for the data set considered $\bar{\varepsilon}^w_\alpha(\hat{F}_n)$ is decreasing. Table 5 presents the bootstrap estimates of the bias and the mean-squared error for $\bar{\varepsilon}^w_\alpha(X;t)$ with samples of size $n = 67$.

6.2. Simulation Studies

In this section, we carry out a Monte-Carlo simulation study to examine the performance of the empirical estimator $\bar{\varepsilon}^w_\alpha(\hat{F}_n)$. For this, we generate samples from an exponential distribution with parameter value 2. In Figure 3, the dark line represents the theoretical value of $\bar{\varepsilon}^w_\alpha(X,t)$, and the dotted line represents the estimate value of $\bar{\varepsilon}^w_\alpha(\hat{F}_n)$ for various values of t. From Figure 3, it is easy to see that $\bar{\varepsilon}^w_\alpha(\hat{F}_n)$ is decreasing in t. In Table 6, we present the bias and mean-squared error (MSE) of $\bar{\varepsilon}^w_\alpha(\hat{F}_n)$ for various values of t $(0.01 \leq t \leq 2)$ and sample sizes 50 and 100. It is clear from the table that as the sample size increases, absolute values of bias and MSE decreases.

Table 6. Bias and MSE of $\bar{\varepsilon}^w_\alpha(\hat{F}_n)$ for $n = 50$ and $n = 100$

	Bias and MSE of $\bar{\varepsilon}^w_\alpha(\hat{F}_n)$			
	$n = 50$		$n = 100$	
t	Bias	MSE	Bias	MSE
0.5	−0.0058	0.0079	−0.0014	0.0072
0.8	−0.0092	0.0053	−0.0052	0.0039
1.0	−0.0069	0.0032	0.0021	0.0029
1.05	0.0067	0.0041	−0.0029	0.0011
1.15	−0.0083	0.0032	0.0067	0.0018
1.25	0.0079	0.0034	0.0022	0.0021
1.45	0.0006	0.0047	−0.0002	0.0012
1.50	−0.0088	0.0059	−0.0063	0.0018
1.60	0.0023	0.0036	−0.0018	0.0022
1.65	0.0088	0.0019	0.0058	0.0004
1.75	−0.0095	0.0009	−0.0003	0.0007
1.85	−0.0079	0.0034	0.0053	0.0011

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